UNIVERSITY OF PENNSYLVANIA

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CLEMENT WINSTON

A THESIS

IN MATHEMATICS

PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

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ON MECHANICAL QUADRATURES FORMULAE INVOLVING THE CLASSICAL ORTHOGONAL POLYNOMIALS[1]

BY C. WINSTON

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Introduction. Let $\psi(x)$ be a function bounded and non-decreasing, with infinitely many points of increase, over an interval (a, b), finite or infinite, and such that all "moments" $\alpha_n = \int_a^b x^n d\psi(x)$ exist $(n = 0, 1, \cdots)$, with $\alpha_0 > 0$. It gives rise, as is known, to an infinite sequence of orthogonal and normal Tchebycheff polynomials

$$\varphi_n(x; a, b; d\psi(x)) \equiv \varphi_n(x; d\psi) \equiv \varphi_n(x) \equiv a_n(d\psi)\phi_n(x; d\psi),$$

(1)

$$\phi_n(x; d\psi) \equiv \phi_n(x) \equiv x^n - S_n x^{n-1} + \cdots \qquad (a_n(d\psi) \equiv a_n > 0; n = 0, 1, \cdots).$$

The sequences $\{\varphi_n(x; d\psi)\}$, $\{\phi_n(x; d\psi)\}$ are uniquely determined by the respective relations

$$\int_a^b \varphi_n(x)\varphi_m(x)d\psi(x) = \delta_{nm} \qquad (n, m = 0, 1, \cdots),$$

(2)

$$\int_a^b \phi_n(x)G_{n-1}(x)d\psi(x) = 0 \qquad (n = 1, 2, \cdots),$$

where hereafter $G_s(x)$ generally denotes an arbitrary polynomial of degree $\leqq s$.

A special case of (1) is derived by letting $d\psi(x) = p(x)dx$, $p(x) \geqq 0$ in (a, b). Here we use the notations

$$\varphi_n(x; a, b; p(x)) \equiv \varphi_n(x; p), \quad \phi_n(x; a, b; p(x)) \equiv \phi_n(x; p), \quad a_n(p), \quad S_n(p), \cdots.$$

One of the most important properties of the polynomials (1) is that the zeros $\{x_{in,}\}$ of $\phi_n(x)$ are all real, distinct and lie in (a, b):

$$a < x_{1,n} < x_{2,n} < \cdots < x_{n,n} < b \qquad (x_{i,n}(a, b; d\psi(x)) \equiv x_{i,n}(d\psi) \equiv x_{i,n}).$$

Making use of the Lagrange interpolation polynomial coinciding in value with a given function $f(x)$ at the above points $\{x_{i,n}\}$, we construct a formula of mechanical quadratures of Gauss' type:

[1] The author wishes to express his gratitude to Professor J. Shohat for his valuable suggestions and very helpful criticism.

$$(3) \qquad \int_a^b f(x)\,d\psi(x) = \sum_{i=1}^{n} H_{i,\,n} f(x_{i,\,n}) + R_n(f)$$

$$\left(H_{i,\,n} = \int_a^b \frac{\phi_n(x)\,d\psi(x)}{(x - x_{i,\,n})\,\phi_n'(x_{i,\,n})}\right),$$

with the remarkable property (a ready consequence of (2)):

$$\int_a^b G_{2n-1}(x)\,d\psi(x) = \sum_{i=1}^{n} H_{i,\,n} G_{2n-1}(x_{i,\,n}), \qquad \text{i.e.,} \qquad R_n(G_{2n-1}(x)) \equiv 0.$$

We call (3) of Gauss' type since Gauss [1][2] was first to consider (3) for the simplest case: $\psi(x) \equiv x$, (a, b) finite, $\varphi_n(x)$ being the Legendre polynomial.

The sequence (1) is said to be "symmetric," if

$$(a, b) \equiv (-h, h),\ \psi(x) \equiv -\psi(-x) \text{ (or } p(x) \equiv p(-x)).$$

It follows readily from (2, 3) that here, for any n,

$$(4) \qquad \begin{aligned} &\phi_n(x; d\psi) \equiv (-1)^n \phi_n(-x; d\psi); \qquad x_{i,\,n} + x_{n+1-i,\,n} = 0; \\ &\qquad\qquad H_{i,\,n} = H_{n+1-i,\,n}; \qquad (i = 1, 2, \cdots, n) \\ &\phi_{2n}(x; -h, h; d\psi) \equiv \phi_n(x^2; 0, h^2; d\psi_1); \qquad \phi_{2n+1}(x; -h, h; d\psi) \\ &\qquad\qquad \equiv x\phi_n(x^2; 0, h^2; d\psi_2); \\ &(d\psi_1(x) \equiv 2d\psi(\sqrt{x}), \qquad d\psi_2(x) \equiv x d\psi_1(x)). \end{aligned}$$

From (3, 4) we get at once:

$$(5) \qquad \begin{aligned} &x_{i,\,2n}(-h, h; d\psi) = -x_{2n+1-i,\,2n}(-h, h; d\psi) = x_{i-n,\,n}(0, h^2; d\psi_1), \\ &\qquad\qquad (i = n + 1, n + 2, \cdots, 2n) \\ &H_{i,\,2n}(-h, h; d\psi) = H_{2n+1-i,\,2n}(-h, h; d\psi) = \tfrac{1}{2} H_{i-n,\,n}(0, h^2; d\psi_1), \\ &x_{i,\,2n+1}(-h, h; d\psi) = -x_{2n+2-i,\,2n+1}(-h, h; d\psi) = x_{i-n-i,\,n}(0, h^2; d\psi_2), \\ &\qquad\qquad (i = n + 2, \cdots, 2n + 1) \\ &H_{i,\,2n+1}(-h, h; d\psi) = H_{2n+2-i,\,2n+1}(-h, h; d\psi) = \frac{H_{i-n-1,\,n}}{2x_{i-n-1,\,n}}\,(0, h^2; d\psi_2). \end{aligned}$$

In connection with formula (3) there naturally arises the question of convergence: for what functions $f(x)$ is $\lim_{n\to\infty} R_n(f) = 0$? For (a, b) finite and $d\psi(x) = p(x)dx$ this was almost completely answered by Stieltjes [2]. In recent years the case of (a, b) infinite was treated by several writers [3, 4].

It is evident that the convergence of (3) is intimately related to two questions: (i) how do the coefficients $\{H_{i,\,n}\}$ behave for n very large? (ii) how are the zeros $\{x_{i,\,n}\}$ distributed over (a, b) for n very large?

[2] The numbers in [ ] refer to the bibliography at the end of the paper.

The object of this paper is a study of questions (i) and (ii) for the case of the so-called classical orthogonal polynomials—of Jacobi, Laguerre and Hermite:

$$\text{J: } d\psi(x) = (x-a)^{\alpha-1}(b-x)^{\beta-1}\,dx, \qquad (a, b) \text{ finite}, \qquad \alpha, \beta > 0.$$

(6)[3] $$\text{L: } d\psi(x) = x^{\alpha-1}e^{-x}\,dx, \qquad (a, b) \equiv (0, \infty), \qquad \alpha > 0.$$

$$\text{H: } d\psi(x) = e^{-x^2}\,dx, \qquad (a, b) \equiv (-\infty, \infty).$$

From (4, 5, 6) we note the following relations between Hermite and Laguerre polynomials:

$$\phi_{2n}(x; -\infty, \infty; e^{-x^2}) \equiv \phi_n(x^2; 0, \infty; x^{-\frac{1}{2}}e^{-x});$$

$$\phi_{2n+1}(x; -\infty, \infty; e^{-x^2}) \equiv x\phi_n(x^2; 0, \infty; x^{\frac{1}{2}}e^{-x});$$

$$x_{i,\,2n}(-\infty, \infty; e^{-x^2}) = -x_{2n+1-i,\,2n}(-\infty, \infty; e^{-x^2}) = x_{i-n,\,n}(0, \infty; x^{-\frac{1}{2}}e^{-x})$$

$$(i = n+1, \cdots, 2n).$$

(7) $$H_{i,\,2n}(-\infty, \infty; e^{-x^2}) = H_{2n+1-i,\,2n}(-\infty, \infty; e^{-x^2}) = \tfrac{1}{2}H_{i-n,\,n}(0, \infty; x^{-\frac{1}{2}}e^{-x}),$$

$$x_{i,\,2n+1}(-\infty, \infty; e^{-x^2}) = -x_{2n+2-i,\,2n+1}(-\infty, \infty; e^{-x^2}) = x_{i-n-1,\,n}(0, \infty; x^{\frac{1}{2}}e^{-x}),$$

$$(i = n+2, \cdots, 2n+1)$$

$$H_{i,\,2n+1}(-\infty, \infty; e^{-x^2}) = H_{2n+2-i,\,2n+1}(-\infty, \infty; e^{-x^2}) = \frac{H_{i-n-1,\,n}}{2x_{i-n-1,\,n}}(0, \infty; x^{\frac{1}{2}}e^{-x}).$$

It is generally known that all the coefficients $H_{i,\,n} \to 0$ as $n \to \infty$. Furthermore, Sonine [5], in the case of Hermite polynomials, was able to show, by a very ingenious analysis, that

$$H_{i,\,n} < \frac{\pi}{\sqrt{2n \mp 1}}, \qquad \left(\mp \text{ for } n \begin{matrix}\text{even}\\ \text{odd}\end{matrix}; i = 1, 2, \cdots, n\right).$$

Regarding the zeros $\{x_{i,\,n}\}$ we know they are everywhere dense in every finite part of (a, b), with $x_{1,\,n} \to a$, $x_{n,\,n} \to b$, $(n \to \infty)$. E. R. Neumann [6] has shown, for Laguerre polynomials proper, i.e., with $\alpha = 1$, by means of interesting geometrical considerations, that

$$x_{i,\,n} = C_{i,\,n}\frac{(i+1)^2}{n+1}; \qquad \tfrac{1}{4} < C_{i,\,n} < 4 \qquad (i = 1, 2, \cdots, n).$$

In what follows, regarding question (i), we extend Sonine's method so as not only to embrace all classical polynomials, but also to obtain both upper and lower bounds for the coefficients $\{H_{i,\,n}\}$, their respective order of magnitude, their order with respect to n, and, in some cases, their asymptotic expression for $n \to \infty$. Regarding question (ii), we extend Neumann's method so as to include the general case of Laguerre polynomials for any $\alpha(> 0)$, whence the case of Hermite polynomials follows at once (by (7)).

[3] J, L, H stand, hereafter, for Jacobi, Laguerre and Hermite polynomials respectively.

This paper also contains a generalization of the mechanical quadratures formula where some of the points, at which the value of the function $f(x)$ to be integrated is known, are fixed.

1. **Some properties of the classical orthogonal polynomials.** These will be needed in our discussion.

$$
\begin{aligned}
&A(x)\phi_n''(x) + B(x)\phi_n'(x) + C_n\phi_n(x) = 0 && (n > 0).\\
&A(x) = hx^2 + kx + l, \qquad B(x) = tx + u. && (h, k, l, t, u, C_n \text{ const.})\\
&\phi_{n+1}(x) = (x - c_{n+1})\phi_n(x) - \lambda_{n+1}\phi_{n-1}(x). && (n \geqq 1)\\
&c_{n+1} = S_{n+1} - S_n; \qquad \lambda_{n+1} = \frac{a_{n-1}^2}{a_n^2}. && (n > 0)
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
&\text{H: } A(x) \equiv 1; \qquad B(x) \equiv -2x; \qquad C_n = 2n; \qquad \lambda_{n+1} = n/2.\\
&\text{L: } A(x) \equiv x; \qquad B(x) \equiv \alpha - x; \qquad C_n = n; \qquad \lambda_{n+1} = n(n + \alpha - 1).\\
&\text{J: } A(x) \equiv 1 - x^2; \qquad B(x) \equiv \alpha - \beta - (\alpha + \beta)x; \qquad C_n = n(n + \alpha + \beta - 1);\\
&\lambda_{n+1} = \frac{4n(n + \alpha + \beta - 2)(n + \alpha - 1)(n + \beta - 1)}{(2n + \alpha + \beta - 1)(2n + \alpha + \beta - 2)^2(2n + \alpha + \beta - 3)}.
\end{aligned} \tag{9}
$$

(Hereafter for J: $(a, b) \equiv (-1, 1)$).

The following relation is fundamental for our discussion:

$$\phi_n'(x; p) \equiv n\phi_{n-1}(x; Ap) \qquad (n \geqq 1; A \equiv A(x) \text{ in (8)}). \tag{10}$$

Hereafter we shall use the abbreviated notations:

$$\phi_n(x; p) \equiv \phi_n(x), \qquad \phi_n(x; Ap) \equiv \phi_n^*(x), \qquad a_n(p) \equiv a_n, \qquad a_n(Ap) \equiv a_n^*, \cdots.$$

Other necessary formulae are:

$$H_{i,n}{}^{[7]} = \frac{1}{a_{n-1}^2\phi_{n-1}(x_{i,n})\phi_n'(x_{i,n})} \qquad (i = 1, 2, \cdots, n). \tag{11}$$

$$
\begin{aligned}
&\sum_{i=1}^{n} H_{i,n} = \sqrt{\pi}, \qquad \phi_{2n}(0) = (-1)^n \frac{(2n)!}{2^{2n}\, n!},\\
&\phi_{2n+1}(0) = 0, \qquad a_n^2 = \frac{2^n}{n!\sqrt{\pi}}.\\
&\phi_{n-1}^{(x)}{}^{[8]} = \frac{(n-1)!\, e^{\frac{n}{2} + \frac{x^2}{2}}}{2^{\frac{n-1}{2}}\sqrt{\pi}\, n^{\frac{n}{2}}\sqrt[4]{1 - \frac{x^2}{2n}}}\left\{\sin\left(\frac{n\pi}{2} - x\sqrt{2n}\right) + 0\left(\frac{1}{\sqrt{n}}\right)\right\}\\
&\qquad\qquad x^2 < 2n\text{-}n^{\frac{1}{3}+\epsilon}, \epsilon > 0
\end{aligned} \tag{12}
$$

H:

$$(13)\ \text{L}:\begin{cases}\displaystyle\sum_{i=1}^{n} H_{i,n}=\Gamma(\alpha);\quad \phi_n^2(0)=\left[\frac{\Gamma(n+\alpha)}{\Gamma(\alpha)}\right]^2;\quad a_n^2=\frac{1}{\Gamma(n+1)\Gamma(n+\alpha)};\\ \phi_n(x)\,[9]=\dfrac{(-1)^n}{\sqrt{\pi}}\,n!\,x^{\frac{1}{4}-\frac{\alpha}{2}}e^{\frac{x}{2}}\,n^{\frac{\alpha}{2}-\frac{3}{4}}\left\{\cos\left(2\sqrt{xn}+\frac{\pi}{4}-\frac{\pi\alpha}{2}\right)+\frac{x^{\frac{3}{2}}}{\sqrt{n}}\right\}\\ 0<a<x<kn^{\frac{1}{3}}\ (a,k\ \text{const.})\end{cases}$$

$$(14)\ \text{J}:\begin{cases}\phi_n(1)=\dfrac{2^n\Gamma(n+\beta)\Gamma(n+\alpha+\beta-1)}{\Gamma(\beta)\Gamma(2n+\alpha+\beta-1)};\\ \phi_n(-1)=\dfrac{(-1)^n\,2^n\,\Gamma(n+a)\Gamma(n+\alpha+\beta-1)}{\Gamma(\alpha)\Gamma(2n+\alpha+\beta-1)};\\ \phi_{2n}(0)=\dfrac{(-1)^n\,(2n)!}{2^n\,n!\,(2\beta+4n-3)(2\beta+4n-5)\cdots(2\beta+2n-1)}\quad(\alpha=\beta);\\ \varphi_n(x)\,[10]=(-1)^n\sqrt{\dfrac{(2n+\alpha+\beta-1)\,\Gamma(n+\alpha+\beta-1)\,\Gamma^2(n+\frac{1}{2})}{\pi 2^{\alpha+\beta-1}\,\Gamma(n+1)\,\Gamma(n+\alpha)\,\Gamma(n+\beta)}}\\ \quad\left\{(\sin\theta)^{\frac{1}{2}-\alpha}\,(\cos\theta)^{\frac{1}{2}-\beta}\left[1-\dfrac{(2\alpha-1)\,(2\beta-1)}{4(2n-1)}\right]\right.\\ \quad\times\cos\left[(2n+\alpha+\beta-1)\,\theta-\pi/4\,(2\alpha-1)\right]\\ \quad-\sin\left[(2n+\alpha+\beta-1)\,\theta-\pi/4\,(2\alpha-1)\right]\\ \times\left[\dfrac{(2\alpha-1)\,(2\alpha-3)}{8(2n-1)}\cot\theta-\dfrac{(2\beta-1)\,(2\beta-3)}{8(2n-1)}\tan\theta\right]+0\left(\dfrac{1}{n^2}\right)\Bigg\}\\ \left(\sin^2\theta=\dfrac{1+x}{2}\right).\end{cases}$$

2. **The behavior of the coefficients** $H_{i,n}$. We transform (11) by means of (10) as follows. Write

$$A(x)\,\varphi_n^*(x)=\sum_{i=0}^{n+2}B_i\,\varphi_i(x)\qquad(A(x)\ \text{given in (8)});$$

$$(15)\quad B_i=0\ (0\leqq i\leqq n-1);\qquad B_n=\frac{a_n}{a_n^*};\qquad B_{n+1}=\frac{a_{n+1}}{a_n^*}\,(S_{n+1}^*-S_{n+1});$$

$$B_{n+2}=\frac{ha_n^*}{a_{n+2}}\ (\text{by (2, 8, 9)}).$$

Substituting these values in (15) and using (8, 10), we get:

$$(16)\quad \phi_{n-1}(x_{i,n})=\frac{A(x_{i,n})\,a_{n-1}^{*2}\,\phi_n'(x_{i,n})}{n(a_{n-1}^2-ha_{n-1}^{*2}\,\lambda_{n+1})};\qquad H_{i,n}=\frac{n(a_{n-1}^2-ha_{n-1}^{*2}\,\lambda_{n+1})}{a_{n-1}^2\,a_{n-1}^{*2}\,A(x_{i,n})\,\phi_n'(x_{i,n})}$$

$$(i=1,2,\cdots,n).$$

Following Sonine, we write

$$H_{i,n} = \frac{n(a_{n-1}^2 - ha_{n-1}^{*2}\lambda_{n+1})}{a_{n-1}^2\, a_{n-1}^{*2}\,(A(x_{i,n})\,\phi_n'(x_{i,n}) + \sigma(x_{i,n})\,\phi_n(x_{i,n}))} \tag{17}$$

$$\equiv \frac{n(a_{n-1}^2 - ha_{n-1}^{*2}\lambda_{n+1})}{a_{n-1}^2\, a_{n-1}^{*2}\,\gamma(x_{i,n})},$$

$$\gamma(x) = A(x)\,\phi_n'^2(x) + \sigma(x)\,\phi_n(x), \tag{18}$$

where $\sigma(x)$ is a certain function of x, finite at $x = x_{i,n}$ for all n and $i = 1, 2, \cdots, n$, and so chosen that $\gamma'(x)$ *be divisible by* $\phi_n'^2(x)$. Differentiate (18) and make use of (8):

$$\gamma'(x) = \phi_n'^2(x)\,[A'(x) - 2B(x)] - 2C_n\phi_n(x)\phi_n'(x) + [\sigma(x)\phi_n(x)]'.$$

Consequently, $\gamma'(x)$ will certainly be divisible by $\phi_n'^2(x)$ if

$$[\sigma(x)\phi_n(x)]' = 2C_n\phi_n(x)\phi_n'(x), \qquad \sigma(x) = C_n\phi_n(x) + \frac{k}{\phi_n(x)} \qquad (k = \text{const.}).$$

Here k must be zero ($\sigma(x_{i,n})$ being finite) so that finally

$$\gamma(x) = A(x)\phi_n'^2(x) + C_n\phi_n^2(x). \tag{19}$$

We have further, in view of (8),

$$\gamma'(x) = 2(h - t)\phi_n'^2(x)\left[x - \frac{2u - k}{2(h - t)}\right] \qquad (h \neq t), \tag{20}$$

$$\gamma'(x) = \phi_n'^2(x)\,(k - 2u) \qquad (h = t) \tag{21}$$

(20, 21) lead directly to the following conclusions.

(i) $h \neq t$. $\gamma(x)$ is minimum or maximum at

$$x = x_0 = \frac{2u - k}{2(h - t)}$$

according as $h > t$ or $h < t$, and $\gamma\left(\dfrac{2u - k}{2(h - t)}\right)$ gives correspondingly an upper or lower bound for $H_{i,n}$.

(ii) $h = t$. This case is possible only for Jacobi polynomials with $\alpha + \beta = 1$ (see (9)). For $k > 2u$, $k < 2u$, $k = 2u$, $\gamma(x)$ is correspondingly monoton increasing, monoton decreasing, or constant. Hence,

$$k > 2u\colon H_{1,n} > H_{2,n} > \cdots > H_{n,n}.$$

$$k < 2u\colon H_{1,n} < H_{2,n} < \cdots < H_{n,n}. \tag{22}$$

$$k = 2u\colon H_{1,n} = H_{2,n} = \cdots = H_{n,n} = \frac{1}{n}\int_{-1}^{+1}\frac{dx}{\sqrt{1 - x^2}} = \frac{\pi}{n}.$$

The constancy of $H_{i,n} (i = 1, 2, \cdots, n)$ is possible for trigonometric polynomials only ($\alpha = \beta = \frac{1}{2}$), as is seen by (9, 22).

We now apply (19–21) to the polynomials J, L, H separately, making use of (12–14).

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$$(23)\quad \gamma(x) = \phi_n'^2(x) + 2n\phi_n^2(x), \quad h > t, \quad x_0 = \frac{2u-k}{2(h-t)} = 0, \quad \gamma(x_0) \text{ is minimum,}$$

$$H_{i,n} \leqq \frac{n}{a_{n-1}^2(\phi_n'^2(0) + 2n\phi_n^2(0))} \qquad (i = 1, 2, \cdots, n).$$

$$H_{i,2m} = H_{2m+1-i,2m} \leqq \frac{(m!)^2 2^{2m}\sqrt{\pi}}{2m(2m!)} = \frac{\pi\Gamma(m)}{2\Gamma(m+\frac{1}{2})} < \frac{\pi}{\sqrt{4m-1}} \; (i = 1, 2, \cdots, 2m),$$

(24)[4]

$$H_{i,2m+1} = H_{2m+2-i,2m+i} \leqq \frac{(m!)^2 2^{2m}\sqrt{\pi}}{(2m+1)!} = \frac{\pi\Gamma(m+1)}{2\Gamma(m+\frac{3}{2})} < \frac{\pi}{\sqrt{4m+3}} \; (i = 1, 2, \cdots, 2m+1).$$

Formulae (24) represent Sonine's results. The above considerations give even more:

$$H_{1,2m} < H_{2,2m} < \cdots < H_{m,2m} = H_{m+1,2m} > H_{m+2,2m} > \cdots > H_{2m,2m}$$

(since $x_{1,2m} < x_{2,2m} < \cdots < x_{m,2m} < 0 < x_{m+1,2m} < \cdots < x_{2m,2m}$). For $n = 2m + 1$ we have similarly:

$$H_{1,2m+1} < H_{2,2m+1} < \cdots < H_{m+1,2m+1} > \cdots > H_{2m+1,2m+1}.$$

$\gamma(x)$ being here an even function of x, it follows by (20, 23) that

$$(25)\quad s < |x_{i,n}| < t \text{ implies } \gamma(s) < \gamma(x_{i,n}) < \gamma(t); \quad \frac{n}{a_{n-1}^2\gamma(t)} < H_{i,n} < \frac{n}{a_{n-1}^2\gamma(s)};$$

$$(26)\quad \frac{\pi}{\sqrt{2n}\, e^{t^2}(1 + O(1/\sqrt{n}))} < H_{i,n} < \frac{\pi}{\sqrt{2n}\, e^{s^2}(1 + O(1/\sqrt{n}))}$$

$$(0 \leqq s < |x_{i,n}| < t < \sqrt{2n - n^{\frac{1}{3}+\epsilon}}\ \epsilon > 0).$$

It will be shown later (64, 65) that

$$\frac{i-m}{2\sqrt{m+1}} < -x_{2m+1-i,2m} = x_{i,2m} < \frac{2(i-m+1)}{\sqrt{m+1}}$$

$$(i = m+1, m+2, \cdots, 2m),$$

$$(27)\quad \frac{i-m}{2\sqrt{m+1}} < -x_{2m+2-i,2m+1} = x_{i,2m+1} < \frac{2(i-m+1)}{\sqrt{m+1}}$$

$$(i = m+2, \cdots, 2m+1).$$

[4] We use here the inequality [5]: $\dfrac{\Gamma(x)}{\Gamma(x+\frac{1}{2})} < \dfrac{2}{\sqrt{4x-1}}$ $(x > \frac{1}{4})$.

Hence, by (26),

$$\frac{\pi}{2\sqrt{m}\, e^{\frac{4(i-m+1)^2}{m+1}}(1+O(1/\sqrt{m}))} < H_{2m+1-i,\,2m}$$

$$= H_{i,\,2m} < \frac{\pi}{2\sqrt{m}\, e^{\frac{(i-m)^2}{4(m+1)}}(1+O(1/\sqrt{m}))} \qquad (i = m+1, \cdots, 2m),$$

(28)

$$\frac{\pi}{2\sqrt{m}\, e^{\frac{4(i-m+1)^2}{m+1}}(1+O(1/\sqrt{m}))} < H_{2m+2-i,\,2m+1}$$

$$= H_{i,\,2m+1} < \frac{\pi}{2\sqrt{m}\, e^{\frac{(i-m)^2}{4(m+1)}}(1+O(1/\sqrt{m}))} \quad (i = m+2, \cdots, 2m+1).$$

From (28) we get an interesting asymptotic expression for some of the $H_{i,\,n}$:

$$(29)\quad \lim_{m\to\infty} \sqrt{m}\, H_{i,\,2m+\sigma} = \pi/2, \qquad \text{i.e.,} \qquad H_{2m+\sigma+1+i,\,2m+\sigma} = H_{i,\,2m+\sigma} = \frac{\pi/2 + o(1)}{\sqrt{m}}$$

$$(\sigma = 0, 1;\ m - m^{\frac{1}{2}-\epsilon} < i < m + m^{\frac{1}{2}-\epsilon},\ \epsilon > 0).$$

Furthermore, the inequalities (28) enable us, in each case where the behavior of $f(x)$ for large $|x|$ is known, to estimate how much the mechanical quadratures formula (3) is affected if we omit even infinitely many of the terms $H_{i,\,n} f(x_{i,\,n})$ corresponding to distant $x_{i,\,n} (\gtrless 0)$.

Illustration. (i) Consider the mechanical quadratures formula (3) for $n = 2m + \sigma$ $(\sigma = 0, 1)$ in which $f(x)$ is such that

$$|f(x)| = O(|x|^l) \qquad (|x| > A \text{ which is sufficiently large}).$$

Then we can omit in (3) all terms $H_{i,\,2m+\sigma} f(x_{i,\,2m+\sigma})$ corresponding to

$$(30)\quad i > m + \sqrt{km \log m}, \qquad i < m - \sqrt{km \log m}$$

$$(k > \text{max. } (0,\ 2\,(l+1))).$$

(k here and hereafter generally denotes a properly fixed positive quantity.) In fact, for all such i, $|x_{i,\,2m+\sigma}| \to \infty$ as $m \to \infty$ (by (27)). Hence, for m sufficiently large, all such $|x_{i,\,2m+\sigma}| > A$. On the other hand, since

$$|x_{i,\,2m+\sigma}| < \sqrt{4m + 2\sigma + 1}$$

(see (53, 61))

$$(31)\quad |\Sigma H_{i,\,2m+\sigma} f(x_{i,\,2m+\sigma})| = O\left(\frac{1}{m^{\frac{k}{4} - \frac{l+1}{2}}}\right) \to 0 \qquad (m \to \infty\,;\ i \text{ and } k \text{ as in (30)}).$$

We see that *the number of terms actually used in* (3) *is only an infinitely small part of the total number of terms therein.*

(ii) Similar considerations show that if

$$| f(x) | = O(e^{|x|^l}) \qquad (l < 1; |x| > A),$$

we can omit in (3), without impairing the convergence, all terms

$$H_{i,\,2m+\sigma} f(x_{i,\,2m+\sigma})$$

for which $i > m + m^{\frac{1}{2}+\epsilon}$, $i < m - m^{\frac{1}{2}+\epsilon}$, $l/2 \leqq \epsilon < \frac{1}{2}$. Here the number of terms which we can afford to omit in (3) again increases indefinitely with m, but is smaller than in the preceding case, which was to be expected, since $| f(x) |$ increases much more strongly at infinity.

LAGUERRE POLYNOMIALS. Here.

$$\gamma(x) = x\phi_n'^2(x) + n\phi_n^2(x), \qquad h > t, \qquad x_0 = \alpha - \tfrac{1}{2}, \qquad \gamma(x_0) \text{ is minimum.} \tag{32}$$

$$H_{i,\,n} \leqq \frac{n}{a_{n-1}^{*2}((\alpha - \frac{1}{2})\,\phi_n'^2(\alpha - \frac{1}{2}) + n\phi_n^2(\alpha - \frac{1}{2}))} \qquad (i = 1, 2, \cdots, n). \tag{33}$$

(i) $\alpha \leqq \frac{1}{2}$.

$$H_{1,\,n} > H_{2,\,n} > \cdots > H_{n,\,n},$$

$$H_{i,\,n} \leqq \frac{1}{a_{n-1}^{*2}\phi_n^2(0)} = \frac{\pi\Gamma(n)}{\Gamma(n + \frac{1}{2})} < \frac{2\pi}{\sqrt{4n-1}} \tag{34}$$

$$(i = 1, 2, \cdots, n) \text{(by (9, 13, 24))}.$$

The last inequality could also have been derived from (7, 24).

(ii) $\alpha > \frac{1}{2}$. As i increases, $H_{i,\,n}$ increases for $0 < x_{i,\,n} \leqq \alpha - \frac{1}{2}$ and decreases for $\alpha - \frac{1}{2} \leqq x_{i,\,n} < \infty$. By (13, 33), we get:

$$H_{i,\,n} < \left(\frac{\alpha - \frac{1}{2}}{e}\right)^{\alpha-\frac{1}{2}} \frac{\pi}{\sqrt{n}\,(1 + O(1/\sqrt{n}))} \qquad (i = 1, 2, \cdots, n), \tag{35}$$

with the same order of magnitude, with respect to n, on the right-hand side as for $\alpha = \frac{1}{2}$. Since here $\gamma(0) > \gamma(\alpha - \frac{1}{2})$, we have, by (13, 33, 35),

$$\frac{\Gamma^2(\alpha)}{(n+\alpha)^\alpha\, e^{\frac{1}{12(n+\alpha)}}} < H_{i,\,n} < \frac{(\alpha - \frac{1}{2})^{\alpha-\frac{1}{2}}\,\pi}{e^{\alpha-\frac{1}{2}}\sqrt{n}\,(1 + O(1/\sqrt{n}))} \quad (0 < x_{i,\,n} < \alpha - \tfrac{1}{2};\ \alpha > \tfrac{1}{2}).$$

Furthermore,

$$\frac{n}{a_{n-1}^{*2}\gamma(t)} < H_{i,\,n} < \frac{n}{a_{n-1}^{*2}\gamma(s)} \qquad (0 < \alpha - \tfrac{1}{2} < s < x_{i,\,n} < t),$$

$$\frac{\pi t^{\alpha-\frac{1}{2}}}{e^t\sqrt{n}\,(1 + O(1/\sqrt{n}))} < H_{i,\,n} < \frac{\pi s^{\alpha-\frac{1}{2}}}{e^s\sqrt{n}\,(1 + O(1/\sqrt{n}))} \tag{36}$$

$$(\alpha - \tfrac{1}{2} < s < x_{i,\,n} < t < n^{\frac{1}{3}-\epsilon};\ \epsilon > 0) \text{(by (13))}.$$

It will be shown later (59) that

$$\frac{(i+2\sqrt{\alpha}-1)^2}{4(n+1)} < x_{i,n} < \frac{4(i+\alpha)^2}{n+\alpha} \qquad \left(i=1,2,\cdots,n; \quad \alpha > \frac{(1+\sqrt{2})^2}{16}\right).$$

Hence

$$(37) \qquad \frac{\pi\left[\frac{4(i+\alpha)^2}{n+\alpha}\right]^{\alpha-\frac{1}{2}}}{e^{\frac{4(i+\alpha)^2}{n+\alpha}}\sqrt{n}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)} < H_{i,n} < \frac{\pi\left[\frac{(i+2\sqrt{\alpha}-1)^2}{4(n+1)}\right]^{\alpha-\frac{1}{2}}}{e^{\frac{(i+2\sqrt{\alpha}-1)^2}{4(n+1)}}\sqrt{n}\left(1+O\left(\frac{1}{\sqrt{n}}\right)\right)}$$

$$k' n^{\frac{1}{2}} < i < k n^{\frac{3}{2}} \ (k, k' \text{ const.}), \epsilon > 0).$$

If in (37) we let $\alpha = \frac{1}{2}$, we get the asymptotic expression

$$H_{i,n} = \frac{\pi + o(1)}{\sqrt{n}} \qquad (n \to \infty; \quad 0 < i < n^{\frac{1}{2}-\epsilon}, \quad \epsilon > 0).$$

This result could have been derived from (7, 29).

As in the case of Hermite polynomials, we may, under certain conditions, omit infinitely many terms, without impairing the convergence of (3).

Illustration. If $|f(x)| = O(x^l)$ for $x > A > 0$ sufficiently large, we can omit in (3) all terms corresponding to $i > \sqrt{kn \log n}$ ($k > \max.(0, 4l+2)$). We could also obtain results similar to those in (ii) for the Hermite case.

Jacobi Polynomials. Here

$$\gamma(x) = (1-x^2)\phi_n'^2(x) + n(n+\alpha+\beta-1)\phi_n^2(x), \qquad H_{i,n} = \frac{\delta_n}{\gamma(x_{i,n})}$$

$$(38) \qquad (i = 1, 2, \cdots, n)$$

$$\delta_n = \frac{n(a_{n-1}^2 + a_{n-1}^{*2}\lambda_{n+1})}{a_{n-1}^2 a_{n-1}^{*2}}.$$

(i) $h \neq t$. $\gamma(x)$ has an extremum at

$$x_0 = \frac{\alpha-\beta}{\alpha+\beta-1}.$$

Case 1: $h > t$. Possible if and only if $\alpha+\beta > 1$.

a) $\beta \leqq \frac{1}{2}$. Here $x_0 = \dfrac{\alpha-\beta}{\alpha+\beta-1} \geqq 1$ for all α, β under consideration.

$\gamma(x_0)$ is minimum; $\qquad H_{1,n} < H_{2,n} < \cdots < H_{n,n}$.

We have further, since $\dfrac{\delta_n}{\gamma(-1)} < H_{i,n} < \dfrac{\delta_n}{\gamma(1)}$:

$$\frac{2^{\alpha+\beta-1}\Gamma^2(\alpha)}{n^{2\alpha}(1+O(1/n))} < H_{i,n} < \frac{2^{\alpha+\beta-1}\Gamma^2(\beta)}{n^{2\beta}(1+O(1/n))} \qquad (i = 1, 2, \cdots, n).$$

b) $\alpha \leqq \frac{1}{2}$. We derive from a), by interchanging α and β:

$$H_{1,n} > H_{2,n} > \cdots > H_{n,n}; \qquad \frac{2^{\alpha+\beta-1}\Gamma^2(\beta)}{n^{2\beta}(1+O(1/n))} < H_{i,n} < \frac{2^{\alpha+\beta-1}\Gamma^2(\alpha)}{n^{2\alpha}(1+O(1/n))}$$

$$(i = 1, 2, \cdots, n).$$

c) $\alpha > \frac{1}{2}, \beta > \frac{1}{2}, \alpha > \beta$. Here, $0 < x_0 = \dfrac{\alpha-\beta}{\alpha+\beta-1} < 1$, and as i increases, $H_{i,n}$ increases for $-1 < x_{i,n} \leqq x_0$ and decreases for $x_0 \leqq x_{i,n} < 1$. Hence, (by (9, 14)),

$$\frac{\delta_n}{\gamma(-1)} < H_{i,n} < \frac{\delta_n}{\gamma(x_0)}; \qquad \frac{2^{\alpha+\beta-1}\Gamma^2(\alpha)}{n^{2\alpha}(1+O(1/n))} < H_{i,n} < \frac{\pi(1+x_0)^{2\alpha}(1-x_0)^{2\beta}}{2^{\alpha+\beta+1}\, n(1+O(1/n))}$$

$$(-1 < x_{i,n} \leqq x_0);$$

$$\frac{\delta_n}{\gamma(1)} < H_{i,n} < \frac{\delta_n}{\gamma(x_0)}; \qquad \frac{2^{\alpha+\beta-1}\Gamma^2(\beta)}{n^{2\beta}(1+O(1/n))} < H_{i,n} < \frac{\pi(1+x_0)^{2\alpha}(1-x_0)^{2\beta}}{2^{\alpha+\beta-1}\, n(1+O(1/n))}$$

$$(x_0 \leqq x_{i,n} < 1).$$

d) $\alpha > \frac{1}{2}, \beta > \frac{1}{2}, \beta > \alpha$. The results are the same as for c), x_0 here lying in $(-1, 0)$.

e) $\alpha = \beta > \frac{1}{2}$ (Symmetric case). Here $x_0 = 0$, $\gamma(0)$ is minimum;

$$H_{1,2m} < H_{2,2m} < \cdots < H_{m,2m} = H_{m+1,2m} > \cdots > H_{2m,2m}.$$

$$H_{1,2m} = H_{2m,2m} < \frac{2^{2(\beta-1)}\Gamma^2(\beta)}{m\,\Gamma(2\beta)}; \qquad H_{m,2m} = H_{m+1,2m} > \frac{2^{2(\beta-1)}\Gamma^2(\beta)}{m\,\Gamma(2\beta)}.$$

$$H_{1,2m+1} < H_{2,2m+1} < \cdots < H_{m+1,2m+1} > H_{m+2,2m+1} > \cdots > H_{2m+1,2m+1};$$

$$H_{1,2m+1} = H_{2m+1,2m+1} < \frac{2^{2\beta-1}\Gamma^2(\beta)}{(2m+1)\,\Gamma(2\beta)}; \qquad H_{m+1,2m+1} > \frac{2^{2\beta-1}\Gamma^2(\beta)}{(2m+1)\,\Gamma(2\beta)};$$

$$\frac{\Gamma^2(\beta)\,2^{2\beta-1}}{m^{2\beta}\,(1+O(1/m))} < H_{i,2m+\sigma} < \frac{\pi\,2^{2\beta-1}}{m(1+O(1/m))}$$

$$(\sigma = 0, 1; i = 1, 2, \cdots, 2m+\sigma).$$

Case 2. $h < t$. Possible if and only if $\alpha + \beta < 1$.

a) $\beta > \alpha$. $x_0 = \dfrac{\alpha-\beta}{\alpha+\beta-1} \geqq 1$, $\gamma(x_0)$ is maximum;

$$H_{1,n} > H_{2,n} > \cdots > H_{n,n}; \frac{2^{\alpha+\beta-1}\Gamma^2(\alpha)}{n^{2\alpha}\,(1+O(1/n))} > H_{i,n} > \frac{2^{\alpha+\beta-1}\Gamma^2(\beta)}{n^{2\beta}\,(1+O(1/n))}$$

$$(i = 1, 2, \cdots, n).$$

b) $\alpha > \beta$. $x_0 \leqq -1$, $\gamma(x_0)$ is maximum. The results concerning $H_{i,n}$ are the same as in a), with inequality signs reversed.

c) $\alpha = \beta$ (Symmetric Case). $x_0 = 0, \gamma(0)$ is maximum. We get the same results as in case 1 e), with inequality signs reversed throughout.

(ii) $h = t$. Possible if and only if $\alpha + \beta = 1$.

a) $k > 2u$ $(\beta > \alpha)$, b) $k < 2u$ $(\alpha > \beta)$, c) $k = 2u$ $(\alpha = \beta = \frac{1}{2})$.

In a) and b) we get the same results respectively as in cases 1 a) and b). c) gives the trigonometric case discussed above (see (22)).

3. **A mechanical quadrature formula with fixed points.** We now fix m distinct points $\xi_1, \xi_2, \cdots, \xi_k, \eta_{k+1}, \eta_{k+2}, \cdots, \eta_m$ outside the interval (a, b) at which the values of the function $f(x)$ to be integrated are known. Making use of the orthogonality property (2) we readily derive

THEOREM I. *In order to have*

$$\int_a^b G_{2n-m-1}(x)\, d\psi(x) = \sum_{i=1}^{n} H_{i,n}\, G_{2n-m-1}(x_{i,n}),$$

$$x_{1,n} < x_{2,n} < \cdots < x_{k,n} \leqq a < x_{m+1,n} < x_{m+2,n} < \cdots < x_{n,n} \tag{39}$$

$$\leqq x_{k+1,n} < x_{k+2,n} < \cdots < x_{m,n}, \qquad x_{i,n} = \xi_i \ (i = 1, 2, \cdots, k),$$

$$x_{i,n} = \eta_i \ (i = k+1, \cdots, m),$$

where

$$H_{i,n} = \int_a^b \frac{\theta_n(x) d\psi(x)}{(x - x_{i,n})\, \theta_n'(x_{i,n})}, \qquad \theta_n(x) = \prod_{i=1}^{n} (x - x_{i,n}),$$

it is necessary and sufficient that $x_{i,n}$ $(i = m+1, \cdots, n)$ *be the zeros of the Tchebycheff polynomial* $\varphi_{n-m}(x; a, b; d\psi_1(x))$, *with*

$$d\psi_1(x) = \prod_{i=1}^{k} (x - \xi_i) \prod_{i=k+1}^{m} (\eta_i - x)\, d\psi(x).$$

We proceed to derive some relations for the $H_{i,n}$ corresponding to the fixed points. Consider first the interval (a, b), a being finite, with one point fixed at $\xi \leqq a$. Letting $d\psi_1(x) = (x - \xi)\, d\psi(x)$, we have, by the above theorem:

$$H_{i,n} = \int_a^b \frac{(x - \xi)\varphi_{n-1}(x; d\psi_1(x))\, d\psi(x)}{(x - x_{i,n})\, [(x - \xi)\, \varphi_{n-1}(x; d\psi_1(x))]'_{x = x_{i,n}}} \qquad (i = 1, 2, \cdots, n),$$

$$x_{1,n} = \xi \leqq a, \qquad \varphi_{n-1}(x_{i,n}; d\psi_1(x)) = 0 \qquad (i = 2, 3, \cdots, n),$$

$$H_{1,n} = H_\xi = \int_a^b \frac{\varphi_{n-1}^*(x)\, d\psi(x)}{\varphi_{n-1}^*(\xi)}, \qquad (\varphi_n^*(x) \equiv \varphi_n(x; d\psi_1(x)), \text{ with } a_n^*, \cdots).$$

Using the expansion

$$(x - \xi)\, \varphi_{n-1}^*(x) = \frac{a_{n-1}^*}{a_n\, \varphi_{n-1}(\xi)} \frac{\varphi_n(x)\varphi_{n-1}(\xi) - \varphi_n(\xi)\varphi_{n-1}(x)}{(x - \xi)}$$

and the Darboux formulae [10]

$$K_{n-1}(x;\xi;d\psi) \equiv K_{n-1}(x;\xi) \equiv \sum_{i=0}^{n-1} \varphi_i(x)\varphi_i(\xi) = \frac{a_{n-1}}{a_n} \frac{\varphi_n(x)\varphi_{n-1}(\xi) - \varphi_n(\xi)\varphi_{n-1}(x)}{x-\xi}$$

$$K_{n-1}(\xi;d\psi) \equiv K_{n-1}(\xi;\xi;d\psi) \equiv \sum_{i=0}^{n-1} \varphi_i^2(\xi;d\psi) = \cdots,$$

we get

$$H_\xi^{[11]} = \frac{a_{n-1}^*}{a_{n-1}\varphi_{n-1}(\xi)\,\varphi_{n-1}^*(\xi)} \int_a^b \sum_{i=0}^{n-1} \varphi_i(x)\varphi_i(\xi)\,d\psi(x)$$

$$= \frac{a_{n-1}^*}{a_{n-1}\varphi_{n-1}(\xi)\,\varphi_{n-1}^*(\xi)} = \frac{1}{K_{n-1}(\xi;d\psi)}.$$

Similarly, if b is finite and we fix one point $\eta \geqq b$:

$$H_\eta = \frac{1}{K_{n-1}(\eta;d\psi)}.$$

The general case of m fixed points can be dealt with now very readily. With the notations (39), we get, letting $d\psi_1(x) = M(x)\,N(x)d\psi(x)$, with $M(x) \equiv \prod_{i=1}^{k}(x-\xi_i)$, $N(x) \equiv \prod_{i=k+1}^{m}(\eta_i - x)$:

$$H_{i,n} = \int_a^b \frac{M(x)\,N(x)\,\varphi_{n-m}(x;d\psi_1(x))\,d\psi(x)}{(x-x_{i,n})\,[M(x)\,N(x)\,\varphi_{n-m}(x;d\psi_1(x))]'_{x=x_{i,n}}} \qquad (i=1,2,\cdots,n).$$

For $1 \leqq i \leqq k$; $H_{i,n} \equiv H_{\xi_i} = \int_a^b \frac{\frac{M(x)N(x)}{(x-\xi_i)}\varphi_{n-m}(x;d\psi_1(x))d\psi(x)}{M'(\xi_i)N(\xi_i)\varphi_{n-m}(\xi_i;d\psi_1(x))}$

$$= \frac{1}{M'(\xi_i)N(\xi_i)} \int_a^b \frac{\varphi_{n-m}(x;d\psi_1(x))d\psi_{1,i}(x)}{\varphi_{n-m}(\xi_i;d\psi_1(x))} = \frac{1}{M'(\xi_i)N(\xi_i)K_{n-m}(\xi_i;d\psi_{1,i}(x))}$$

$$\left(d\psi_{1,i}(x) = \frac{M(x)N(x)d\psi_1(x)}{x-\xi_i}\right).$$

For $k+1 \leqq i \leqq m$; $\qquad H_{i,n} \equiv H_{\eta_i}$

$$= \int_a^b \frac{\frac{M(x)N(x)}{\eta_i - x}\varphi_{n-m}(x;\,d\psi_1(x))d\psi(x)}{M(\eta_i)N'(\eta_i)\varphi_{n-m}(\eta_i;\,d\psi_1(x))} = \frac{1}{M(\eta_i)N'(\eta_i)} \int_a^b \frac{\varphi_{n-m}(x;d\psi_1(x))d\psi_{1,i}(x)}{\varphi_{n-m}(\eta_i;\,d\psi_1(x))}$$

$$= \frac{1}{M(\eta_i)N'(\eta_i)K_{n-m}(\eta_i;\,d\psi_{1,i}(x))} \left(d\psi_{1,i}(x) = \frac{M(x)N(x)d\psi_1(x)}{(\eta_i - x)}\right).$$

If, for example, (a, b) is finite, $m = 2$, $\xi_1 = a$, $\eta_1 = b$, then

$$H_a = \frac{1}{(b-a)K_{n-2}(a;(b-x)d\psi(x))}, \qquad H_b = \frac{1}{(b-a)K_{n-2}(b;(x-a)d\psi(x)}.$$

4. **The distribution of the zeros of Languerre and Hermite Polynomials.** Following a geometrical method the underlying idea of which is due to E. R. Neumann (for Laguerre Polynomials with $\alpha = 1$) [6], we shall find bounds for the zeros $\{x_{i,n}\}$ of Laguerre polynomials for any $\alpha\ (> 0)$ and incidentally for Hermite polynomials. Hereafter,

$$(40)\qquad L_n(x; 0, \infty; e^{-x}x^{\alpha-1}) \equiv L_n(x) \equiv (-1)^n \varphi_n(x; 0, \infty; e^{-x}x^{\alpha-1})$$

$$\equiv \frac{(-1)^n}{\sqrt{\Gamma(n+1)\Gamma(n+\alpha)}} \sum_{i=0}^{n} (-1)^i \binom{n}{i} \binom{n+\alpha-1}{i} i! x^{n-i}.$$

The following relations, *which hold for any* $\alpha > 0$, are of fundamental importance and may be readily verified using the orthogonality properties of $L_n(x)$:

$$L'_{n+1}(x) - \sqrt{\frac{n+1}{n+\alpha}}\, L'_n(x) = -\sqrt{\frac{n+1}{n+\alpha}}\, L_n(x),$$

$$(41)\qquad (\alpha > 0)$$

$$xL'_{n+1}(x) = (n+1)\, L_{n+1}(x) - \sqrt{(n+1)(n+\alpha)}\, L_n(x).$$

We have further, by (40),

$$\frac{L_n(x)}{L_n(0)} = 1 - \frac{n\Gamma(\alpha)}{\Gamma(\alpha+1)}\, x + \frac{n(n-1)\Gamma(\alpha)}{2!\,\Gamma(\alpha+2)}\, x^2 \pm \cdots + (-1)^n \frac{\Gamma(\alpha)}{\Gamma(\alpha+n)}\, x^n.$$

Hence

$$\frac{L_n(x)}{L_n(0)} > 0 \text{ for } x < \alpha/n,$$

$$(42)\qquad S_n = \sum_{i=1}^{n} x_{i,n} = n(n+\alpha-1),$$

$$(43)\qquad \alpha/n < x_{i,n} < n(n+\alpha-1) \qquad (i = 1, 2, \cdots, n).$$

In what follows $\mu_{i,n}$ and $\mu_{i,n+1}$ represent the abscissas of the extremal points—more briefly, "extremal abscissas"—of the curves $y = L_n(x)$ and $y = L_{n+1}(x)$ respectively. We derive from the differential equation (8, 9) satisfied by Laguerre polynomials:

$$(44)\qquad \Lambda'_n(x) = -n^{\alpha-1} e^{-x} L_n(x) \qquad (\Lambda_n(x) \equiv x^\alpha e^{-x} L'_n(x)).$$

Hence the zeros of $\Lambda_n(x)$ are the extremal abscissas for $y = L_n(x)$ and conversely.

The following relations may be verified, making use of (8, 9, 40, 41):

$$(45)\qquad x\Lambda''_n(x) + (x - \alpha + 1)\,\Lambda'_n(x) + n\Lambda_n(x) = 0,$$

$$(46)\qquad x\sqrt{\frac{n+1}{n+\alpha}}\,\Lambda'_n(x) = n\left(\Lambda_{n+1}(x) - \sqrt{\frac{n+1}{n+\alpha}}\,\Lambda_n(x)\right).$$

$$(47)\qquad \sqrt{(n+1)(n+\alpha)}\,\Lambda'_n(x) - n\Lambda'_{n+1}(x) = n\Lambda_{n+1}(x).$$

$$(48)\quad n\sqrt{\frac{n+\alpha}{n+1}}\,\Lambda_{n+1}(x)-(2n+\alpha-1-x)\,\Lambda_n(x)$$
$$+\sqrt{n(n+\alpha-1)}\,\Lambda_{n-1}(x)=0.$$

From (44, 46) it follows that the curve $y=\Lambda_{n+1}(x)$ intersects $y=\Lambda_n(x)$ at $x=x_{i,n}$ $(i=1,2,\cdots,n)$ (extremal abscissas for the latter curve) if $\alpha=1$, and to the right or left of $x_{i,n}$ according as $\alpha<1$ or $\alpha>1$. Formulae (44, 46, 47) give further:

$$n\int_{x_{i,n+1}}^{x_{i,n}}\Lambda_{n+1}(x)\,dx=\sqrt{(n+1)(n+\alpha)}\,\Lambda_n(x)-n\,\Lambda_{n+1}(x)\Big|_{x_{i,n+1}}^{x_{i,n}}$$
$$=\alpha\Lambda_{n+1}(x_{i,n})+(x_{i,n+1}-\alpha)\,\Lambda_{n+1}(x_{i,n+1}).$$

Let us take i even.[5] Then,

$$\Lambda_n(x_{i,n})\text{ is maximum},\qquad \Lambda_{n+1}(x_{i,n+1})\text{ is maximum};$$

$$\Lambda_n(x_{i,n})>\Lambda_n(x_{i,n+1})=\frac{n+\alpha-x_{i,n+1}}{\sqrt{(n+1)(n+\alpha)}}\Lambda_{n+1}(x_{i,n+1})\text{(by 44)},$$

$$n\int_{x_{i,n+1}}^{x_{i,n}}\Lambda_{n+1}(x)\,dx>\left(\frac{-\alpha x_{i,n+1}}{n+\alpha}+x_{i,n+1}\right)\Lambda_{n+1}(x_{i,n+1}),$$

$$\Lambda_{n+1}(x_{i,n+1})(x_{i,n}-x_{i,n+1})>\int_{n_{i,n+1}}^{x_{i,n}}\Lambda_{n+1}(x)\,dx,$$

$$n(x_{i,n}-x_{i,n+1})>\frac{-\alpha x_{i,n+1}}{n+\alpha}+x_{i,n+1},$$

$$(49)\qquad x_{i,n+1}<\frac{n+\alpha}{n+\alpha+1}\,x_{i,n},$$

$$(50)\qquad x_{i,n}<\frac{\alpha+i}{n+\alpha}\,x_{i,i}<\frac{\alpha+i}{n+\alpha}\,i(i+\alpha-1)\qquad(\alpha>0)\text{(see (42))}.$$

In particular, for $i=1$, and in conjunction with (43):

$$(51)\qquad \alpha/n<x_{1,n}<\alpha/n\,(1+\alpha).$$

We have further, by (42, 49):

$$n(n+\alpha-1)=\sum_{i=1}^{n}x_{i,n}<\frac{n+\alpha-1}{n+\alpha}\sum_{i=1}^{n-1}x_{i,n-1}+x_{n,n}$$
$$=x_{n,n}+\frac{n+\alpha-1}{n+\alpha}(n-1)(n+\alpha-2),$$

$$(52)\qquad x_{n,n}>3n+\alpha-5+\frac{2(\alpha+1)}{n+\alpha}\qquad(\alpha>0).$$

[5] If i is odd, we simply change the sign of $L_n(x)$.

It is known that $x_{n,n} < 4n + 2\alpha$; hence

$$3n + \alpha - 5 < x_{n,n} < 4n + 2\alpha \qquad (\alpha > 0).^{6} \tag{53}$$

In what follows, the figures, which for the sake of brevity have been omitted, are quite similar to those used by Neumann. From (8, 9, 40, 41) it is seen that the abscissa of each point of intersection of $y = L_n(x)$ and $y = L_{n+1}(x)$ is to the right or to the left of $x = \mu_{i,n+1}$ $(i = 1, 2, \cdots, n)$ according as $\alpha > 1$ or $\alpha < 1$. If $\alpha = 1$ we have Neumann's case: viz. the abscissas in question coincide with the points $x = \mu_{i,n+1}$.

Considerations similar to those employed by Neumann lead to the inequalities:

$$x_{1,n+1} < x_{1,n} < \mu_{1,n+1} < \begin{Bmatrix} x_{2,n+1} \\ \mu_{1,n} \end{Bmatrix} < x_{2,n} < \mu_{2,n+1} < \begin{Bmatrix} x_{3,n+1} \\ \mu_{2,n} \end{Bmatrix} < \cdots,$$

where $\mu_{i,n} \lesseqgtr x_{i+1,n+1}$ according as $\mu_{i,n} \lesseqgtr n + \alpha$. We also see that in the particular interval $(x_{i,n}, x_{i+1,n})$ containing α there will be no point of inflection of the curve $y = L_n(x)$. In every other interval $(x_{i,n}, x_{i+1,n})$ there will be just one such point of inflection situated to the right or left of $x = \mu_{i,n+1}$ according as $(x_{i,n}, x_{i+1,n})$ is to the left or right of $x = \alpha$.

Case 1. a) $x_{i+1,n} < \alpha$, $\alpha \leqq 1$. Here the above point of inflection lies between $\mu_{i,n+1}$ and $x_{i+1,n}$. Compare the area

$$\frac{(\mu_{i,n+1} - x_{i,n})^2}{2} L_n'(x_{i,n}) = \frac{(\mu_{i,n+1} - x_{i,n})^2}{2x_{i,n}} \sqrt{(n+1)(n+\alpha)}\, L_{n+1}(x_{i,n}) \quad \text{(by (41))}$$

of the triangle formed by the x-axis, the tangent to $y = L_n(x)$ at $x = x_{i,n}$ and the perpendicular to the x-axis at $x = \mu_{i,n+1}$ with the area under the curve $y = L_n(x)$ between $x_{i,n}$ and $\mu_{i,n+1}$:

$$\int_{x_{i,n}}^{\mu_{i,n+1}} L_n(x)\, dx = L_n(\mu_{i,n+1}) \left(\frac{1-\alpha}{n+1}\right) + \sqrt{\frac{n+\alpha}{n+1}}\, L_{n+1}(x_{i,n}) > \sqrt{\frac{n+\alpha}{n+1}}\, L_{n+1}(x_{i,n}) \qquad \text{(by (41))}.$$

Thus we get:

$$\frac{(\mu_{i,n+1} - x_{i,n})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i,n}} > \sqrt{\frac{n+\alpha}{n+1}}\,; \quad \mu_{i,n+1} - x_{i,n} > \sqrt{\frac{2x_{i,n}}{n+1}}$$

$$x_{i+1,n} - x_{i,n} > \sqrt{\frac{2x_{i,n}}{n+1}} \qquad (i = 1, 2, \cdots, n-1).$$

b) $x_{i,n} > \alpha$, $\alpha \leqq 1$. Here the above mentioned point of inflection lies in $(x_{i,n}, \mu_{i,n+1})$. The triangle considered is formed by the x-axis, the tangent to $y = L_n(x)$ at $x = x_{i+1,n}$ and the perpendicular to the x-axis at $x = \mu_{i,n}$. Its area

$$-\frac{(x_{i+1,n} - \mu_{i,n+1})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i+1,n}} L_{n+1}(x_{i+1,n}) \tag{54}$$

[6] For $\alpha = \frac{1}{2}, \frac{3}{2}$ see (63).

is greater than the corresponding area under the curve $y = L_n(x)$:

$$\int_{\mu_{i,\,n+1}}^{x_{i+1,\,n}} L_n(x)\,dx = \left(\frac{\alpha-1}{n+1}\right) L_n(\mu_{i,\,n+1}) - \sqrt{\frac{n+\alpha}{n+1}}\, L_{n+1}(x_{i+1,\,n}). \tag{55}$$

We now seek to find a relation between $L_n(\mu_{i,\,n+1})$ and $L_{n+1}(x_{i+1,\,n})$. From the location of the above-mentioned point of inflection it is seen that

$$\frac{L_n(\mu_{i,\,n+1})}{x_{i+1,\,n} - \mu_{i,\,n+1}} < -L_n'(x_{i+1,\,n}) = -\frac{\sqrt{(n+1)(n+\alpha)}}{x_{i+1,\,n}} L_{n+1}(x_{i+1,\,n}) \qquad \text{(by (41))},$$

which, combined with (54, 55), leads to

$$-\frac{(x_{i+1,\,n} - \mu_{i,\,n+1})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i+1,\,n}} L_{n+1}(x_{i+1,\,n}) > \int_{\mu_{i,\,n+1}}^{x_{i+1,\,n}} L_n(x)\,dx$$

$$> -\left\{\frac{\alpha-1}{n+1}\frac{(x_{i+1,\,n} - \mu_{i,\,n+1})\sqrt{(n+1)(n+\alpha)}}{x_{i+1,\,n}} + \sqrt{\frac{n+\alpha}{n+1}}\right\} L_{n+1}(x_{i+1,\,n}).$$

Since $L_{n+1}(x_{i+1,\,n}) < 0$ for i even, we get further:

$$(x_{i+1,\,n} - \mu_{i,\,n+1})^2 - \frac{2(\alpha-1)(x_{i+1,\,n} - \mu_{i,\,n+1})}{n+1} > \frac{2x_{i+1,\,n}}{n+1} > \frac{2x_{i,\,n}}{n+1},$$

$$x_{i+1,\,n} - \mu_{i,\,n+1} > \sqrt{\frac{2x_{i,\,n}}{n+1}} - \frac{1-\alpha}{n+1}. \tag{56}$$

Moreover, $L_n(\mu_{i,\,n+1})$ being the greatest value of $L_n(x)$ in $(x_{i,\,n},\ \mu_{i,\,n+1})$, we get (by (41)):

$$L_n(\mu_{i,\,n+1})(\mu_{i,\,n+1} - x_{i,\,n}) > \int_{x_{i,\,n}}^{\mu_{i,\,n+1}} L_n(x)\,dx > L_n(\mu_{i,\,n+1})\left(\frac{1-\alpha}{n+1}\right),$$

$$\mu_{i,\,n+1} - x_{i,\,n} > \frac{1-\alpha}{n+1},$$

and this, added to (56), gives:

$$x_{i+1,\,n} - x_{i,\,n} > \sqrt{\frac{2x_{i,\,n}}{n+1}}. \tag{57}$$

Following Neumann, we now let

$$\rho_i = (n+1)\,x_{i,\,n}, \qquad \sigma_i = \frac{(i + 2\sqrt{\alpha} - 1)^2}{4}.$$

Then

$$\rho_i > \sigma_i \qquad \left(\alpha > \omega = \frac{(1+\sqrt{2})^2}{16} = .364\cdots\right) \qquad \text{(by induction)},$$

$$x_{i,\,n} > \frac{(i + 2\sqrt{\alpha} - 1)^2}{4(n+1)} \qquad (1 \geqq \alpha \geqq \omega;\ i = 1, 2, \cdots, n).$$

To find bounds for $x_{i,\,n}$ for $\alpha \leqq \omega$, let

$$\rho_i' = \frac{(n+1)\omega}{\alpha}\, x_{i,\,n}, \qquad \sigma_i' = \frac{(i + 2\sqrt{\omega} - 1)^2}{4}.$$

Then, as above, $\rho_i' > \sigma_i'$, and

$$x_{i,n} > \frac{(i + 2\sqrt{\omega} - 1)^2}{4(n+1)} \frac{\alpha}{\omega} \qquad (0 < \alpha \leqq \omega; i = 1, 2, \cdots, n).$$

Case 2. a) $x_{i+1,n} < \alpha$, $\alpha \geqq 1$. Here the above point of inflection lies in $(\mu_{i,n+1}, x_{i+1,n})$. In fact, it can be shown to be in $(\mu_{i,n}, x_{i+1,n})$. The area of the triangle formed by the x-axis, the tangent to $y = L_n(x)$ at $x = x_{i,n}$ and the perpendicular to the x-axis at $x = x_{i+1,n+1}$ is greater than $\int_{x_{i,n}}^{x_{i+1,n+1}} L_n(x)\,dx$. i.e., by (41),

$$\frac{(x_{i+1,n+1} - x_{i,n})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i,n}} L_{n+1}(x_{i,n}) > L_n(x_{i+1,n+1}) + \sqrt{\frac{n+\alpha}{n+1}}\, L_{n+1}(x_{i,n}),$$

$$\frac{(x_{i+1,n+1} - x_{i,n})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i,n}} > \sqrt{\frac{n+\alpha}{n+1}} \qquad (\text{since } L_n(x_{i+1,n+1}) > 0),$$

$$x_{i+1,n} - x_{i,n} > \sqrt{\frac{2x_{i,n}}{n+1}} \qquad (x_{i+1,n} < \alpha, \quad \alpha \geqq 1, \quad i = 1, 2, \cdots, n). \tag{58}$$

b) $x_{i,n} > \alpha$, $\alpha \geqq 1$. The point of inflection mentioned above lies in $(x_{i,n}, \mu_{i,n+1})$. The area of the triangle considered is

$$-\frac{(x_{i+1,n} - \mu_{i,n+1})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i+1,n}} L_{n+1}(x_{i+1,n})$$

so that here,

$$-\frac{(x_{i+1,n} - \mu_{i,n+1})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i+1,n}} L_{n+1}(x_{i+1,n}) > \int_{\mu_{i,n+1}}^{x_{i+1,n}} L_n(x)\,dx$$

$$= \left(\frac{\alpha - 1}{n+1}\right) L_n(\mu_{i,n+1}) - \sqrt{\frac{n+\alpha}{n+1}}\, L_{n+1}(x_{i+1,n})$$

$$\frac{(x_{i+1,n} - \mu_{i,n+1})^2 \sqrt{(n+1)(n+\alpha)}}{2x_{i+1,n}} > \sqrt{\frac{n+\alpha}{n+1}} \qquad (\text{by (41)}).$$

Thus we see that *formulae* (57, 58) hold for all $\alpha > 0$. We summarize our results as follows:

$$\frac{(i + 2\sqrt{\alpha} - 1)^2}{4(n+1)} < x_{i,n} < \frac{4(i+\alpha)^2}{n+\alpha} \qquad (\alpha \leqq \omega), \quad (i = 1, 2, \cdots, n) \tag{59}$$

$$\frac{(i + 2\sqrt{\omega} - 1)^2}{4(n+1)} \frac{\alpha}{\omega} < x_{i,n} < \frac{4(i+\alpha)^2}{n+\alpha} \qquad (\alpha \leqq \omega). \tag{60}$$

We now make use of (7), in order to obtain similar results for Hermite polynomials. Let $x_{i,2n+\sigma} (i = 1, 2, \cdots, 2n + \sigma; \sigma = 0, 1)$ represent the zeros of

Hermite polynomials of degree $2n+\sigma$, $x'_{i,n}$, $x''_{i,n}$ $(i=1,2,\cdots,n)$—those of the Laguerre polynomials of degree n, with $\alpha=\frac{1}{2},\frac{3}{2}$ respectively. Then,

$$(61)\qquad \begin{aligned} -x_{2n-1+i,\,2n} &= x_{i,\,2n} = \sqrt{x'_{i-n,\,n}} && (i=n+1,n+2,\cdots,2n),\\ -x_{2n+2-i,\,2n+1} &= x_{i,\,2n+1} = \sqrt{x''_{i-n-1,\,2n}} && (i=n+2,\cdots,2n+1);\end{aligned}$$

$$(62)^{7}\qquad \frac{\sqrt{2}}{2\sqrt{n}} < x_{n+1,\,2n} < \frac{\sqrt{3}}{2\sqrt{n}},\qquad \frac{\sqrt{6}}{2\sqrt{n}} < x_{n+2,\,2n+1} < \frac{\sqrt{15}}{2\sqrt{n}} \qquad \text{(by (5))}.$$

Making use of (53, 59, 60, 61), we get:

$$(63)^{8}\qquad \sqrt{3n-\tfrac{9}{2}} < x_{2n,\,2n} < \sqrt{4n+1},\quad \sqrt{3n-\tfrac{7}{2}} < x_{2n+1,\,2n+1} < \sqrt{4n+3},$$

$$(64)\qquad \begin{aligned} \frac{i-n+\sqrt{2}-1}{2\sqrt{n+1}} < -x_{2n+1-i,\,2n} = x_{i,\,2n} < \frac{2\,(i-n+\frac{1}{2})}{\sqrt{n+\frac{1}{2}}} \\ (i=n+1,\cdots,2n),\\ \frac{i-n+\sqrt{6}-1}{2\sqrt{n+1}} < -x_{2n+2-i,\,2n+1} = x_{i,\,2n+1} < \frac{2\,(i-n+\frac{3}{2})}{\sqrt{n+\frac{3}{2}}} \\ (i=n+2,\cdots,2n+1).\end{aligned}$$

It is interesting to note that results quite similar to those given in (64) may be derived from Neumann's results without first considering the generalized Laguerre polynomials, but by making use of Markoff's theorem [14]. In fact, this theorem yields the inequality

$$\frac{dx_{i,\,n}}{d\alpha} > 0 \qquad (i=1,2,\cdots,n;\ \alpha>0),$$

which enables us to compare $x'_{i,\,n}$ $(\alpha=\frac{1}{2})$, $x''_{i,\,n}$ $(\alpha=\frac{3}{2})$ with $x_{i,\,n}$ $(\alpha=1)$. Thus we get for Hermite polynomials:

$$-x_{2n+1-i,\,2n} = x_{i,\,2n} = \sqrt{x'_{i-n,\,n}} < \sqrt{x_{i-n,\,n}} < \frac{2\,(i-n+1)}{\sqrt{n+1}}$$

$$(i=n+1,\cdots,2n),$$

$$-x_{2n+2-i,\,2n+1} = x_{i,\,2n+1} = \sqrt{x''_{i-n-1,\,n}} > \sqrt{x_{i-n-1,\,n}} > \frac{i-n}{2\sqrt{n+1}}$$

$$(i=n+2,\cdots,2n+1);$$

and since $x_{i,\,2n} > x_{i,\,2n+1}$ $(1 \leqq i \leqq 2n)$,

$$\frac{i-n}{2\sqrt{n+1}} < -x_{2n+1-i,\,2n} = x_{i,\,2n} < \frac{2(i-n+1)}{\sqrt{n+1}} \qquad (i=n+2,\cdots,2n),$$

[7] A. Wiman [12], employing a different method, finds similar results.

[8] The bounds for $x_{n,\,n}$ have been much improved recently by S. C. Van Veen [13] on the basis of a less elementary analysis.

(65)

$$\frac{i-n}{2\sqrt{n+1}} < -x_{2n+2-i,\,2n+1} = x_{i,\,2n+1} < \frac{2(i-n+1)}{\sqrt{n+1}} \quad (i = n+2, \cdots, 2n+1).$$

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Added in Proof (July 9, 1934):

Since the time this paper was sent in for publication there has appeared a paper [*Wolfgang Hahn*—Die Nullstellen der Laguerreschen und Hermiteschen Polynome.—Schriften des Math. Sem. und des Inst. für angewandte Math. der Univ. Berlin, Band 1.] in which results similar to those gotten here for the zeros of Hermite and Laguerre Polynomials are derived by similar methods. No applications to Mechanical Quadratures are made however [cf. also *J. Shohat and C. Winston*; On Mechanical Quadratures, Rend. Del Circ. Mat. Di Palermo, Tomo LVIII, Anno 1934–XII.]